Mathematics of Triple-Junction and Singularity Theory: from CFD of ink-jet printer

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1. Background
2. Motivation: From Ink-jet printer
3. Math of Triple-Junction and Singularity
   3–1. Problem
   3–2. Physics of Triple-Junction
   3–3. Numer. model of Two-phase field
1. Background

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   3-4. Numer. Model of Triple-Junction
Caustics problem

Canon: industrial Math Analysis

Integrable system

Abelian Func. Theory
Emma Previato

Study on Singular strata in Jacobi variety
JMSJ 2007, 2014

Quantized Elastica
John McKay

Today’s topic
Math is of use!

for other math fields: e.g., singularity theory is of use for abelian function theory.

for actual life: e.g., singularity theory is of use for ink-jet printer.

Both are different!
Math is of use for actual life! e.g., for an ink-jet printer.

Leonardo da Vinci said “Mechanics is the paradise of the mathematical sciences because by means of it one comes to the fruits of mathematics.”

In XXI century, we have so many cases that mathematical fields have the strong power for industrial studies. No one predicts what mathematical subject is of use, in this sense, after decades because no one predicts what technology is of use for actual life, basically.
Math is of use for actual life! e.g., for an ink-jet printer.

Leonardo da Vinci said “Mechanics is the paradise of the mathematical sciences because by means of it one comes to the fruits of mathematics.”

It is probabilistic whether some mathematical fields are of use or not. It cannot be predicted!

⇒ Every math field has the possibility but it is important that we should not be too conscious whether our theory becomes of use or not. We study it to be established!

⇒ If a math field is well-established, it will be of use, in this sense, when the time comes.
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   3-4. Numer. Model of Triple-Junction
Purpose of study: to evaluate micro fluid dynamics in an ink-jet printer numerically.

Model (M-Nakano-Shinjo 2011)

Emission device of ink droplets

Numerical computation: emission of ink droplets

Purpose of study: to evaluate micro fluid dynamics in an ink-jet printer numerically.

Emission device of ink droplets

heater

Triple-Junction is Singular!
Triple-phase: Solid, Liquid, Gas!
The singularity of Triple-Junction is the cone \( C\{0,1\} \) or \( C[0,1] \)!

\[
C \{0,1\} \times (0,1) \subseteq C[0,1] \times (0,1)
\]

Today’s topic is the singularity of Triple-Junction!
Mathematical Problem

To construct a mathematical model in order to evaluate fluid dynamics with triple-phase field and triple junctions numerically!

We can assume that these fields are liquids (incompressible fluids).
There was a mathematical model of numerical two-phase fluid dynamics!

Gueyffiler, Li, Nadim, Scardovelli, Zaleski, J.Comp.Phys. 1999
By revising the mathematical model of two-phase fluid dynamics to obtain that of three-phase field!

Discretization

2-phase Euler Equation ↓ 2-phase Difference Euler Equation

Discretization

3-phase Euler Equation ↓ 3-phase Difference Euler Equation

Applied Math (VOF-method)
Arnold, Ebin–Marsden 1970
Diffeomorphism Group
+Momentum Map

1-phase Action Functional
1-phase Euler Equation
1-phase Difference Euler Equation

Applied Math (phase-field theory)

Variation

2-phase Action Functional
2-phase Euler Equation
2-phase Difference Euler Equation

Discretization

3-phase Action Functional
3-phase Euler Equation
3-phase Difference Euler Equation

Applied Math (VOF-method)

Analytic Geometry
Singularity Theory
In order to obtain the three-phase Euler equation, we used **primitive facts in**
1. Diffeomorphism Group
2. Momentum Map
3. Analytic Geometry
Arnold, Ebin–Marsden 1970
Diffeomorphism Group
+Momentum Map

Applied Math (phase–field theory)

Variation
Discretization

1-phase Action Functional
1-phase Euler Equation
1-phase Difference Euler Equation

Discretization

1-phase
Euler Equation

Variation

1-phase
Action Functional
2-phase
Euler Equation
2-phase Difference Euler Equation

Discretization

2-phase
Euler Equation

Variation

1-phase
Action Functional
3-phase
Euler Equation
3-phase Difference Euler Equation

Discretization

3-phase
Euler Equation

Analytic Geometry

Applied Math (VOF–method)

Singularity Theory

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   3-4. Numer. Model of Triple-Junction
Triple-Junction

http://photo-pot.com/?p=23192
Reduced Mathematical Problem

To construct a mathematical model in order to evaluate the triple junctions numerically!
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Shape of Droplet with Triple-Junction

Shape of Droplet is determined so that it minimizes the interface energy under the preservation of volume. (For a small droplet, the gravity is neglected!)
Shape of Droplet with Triple-Junction

\[ E = \sigma_{CA} S_{CA} + \sigma_{AB} S_{AB} + \sigma_{BC} S_{BC} \]

- \( S_{CA} \): Area of Interface between C and A
- \( S_{AB} \): Area of Interface between A and B
- \( S_{BC} \): Area of Interface between B and C
- \( \sigma_{*#} \): positive real number dep. on * and #.

Shape of Droplet is determined so that it minimizes the interface energy under the preservation of volume. (For a small droplet, the gravity is neglected!)
Shape of Droplet with Triple-Junction

\[ E = \sigma_{CA}S_{CA} + \sigma_{AB}S_{AB} + \sigma_{BC}S_{BC} \]

- \( S_{CA} \): Area of Interface between C and A
- \( S_{AB} \): Area of Interface between A and B
- \( S_{BC} \): Area of Interface between B and C
- \( \sigma_{*#} \): Surface tension value of * and #.

Shape of Droplet is determined so that it minimizes the interface energy under the preservation of volume. (For a small droplet, the gravity is neglected!)
Shape of interface (surface tension)

Laplace equation for surface tension

\[ p_1 - p_2 = \sigma \cdot \kappa(x) \]

\( p_1, p_2 \): pressure
\( \sigma \): surface tension value

\( \kappa = \text{div } n \) for the normal unit vector field \( n \)
\( \kappa / 2 \) is the mean curvature,

The constant mean curvature surface is realized.
\[ \sigma_{BC} = \sigma_{AB} + \sigma_{CA} \cos \theta \]

Shape of Droplet with Triple-Junction

\( \theta \): Contact angle

C: Gas

B: Solid

A: Liquid

\( \sigma_{BC} \)

\( \sigma_{AB} \)

\( \sigma_{CA} \)

(The pressures in matters are naturally defined!)
Shape of Droplet with Triple-Junction

\[ E = \sigma_{CA}S_{CA} + \sigma_{AB}S_{AB} + \sigma_{BC}S_{BC} \]

\[ = \sigma_{CA}S_{CA} + \sigma_{AB}S_{AB} + \sigma_{BC} \text{(Const.} - S_{AB}) \]

\[ = \sigma_{CA}S_{CA} + (\sigma_{AB} - \sigma_{BC})S_{AB} + \text{Const.} \]

\[ S_{AB} = \pi r^2 \sin^2 \theta \]

\[ S_{CA} = 2\pi r^2 (1 - \cos \theta) \]

\[ V_A = \frac{4}{3} \pi r^3 (2 + \cos \theta)(1 - \cos \theta)^2 \]
Shape of Droplet with Triple-Junction

\[ E_L = \sigma_{CA} S_{CA} + (\sigma_{AB} - \sigma_{BC}) S_{AB} + \lambda (V_A - V_0) \]

\[ \frac{\delta E_L}{\delta r} = 0, \quad \frac{\delta E_L}{\delta \theta} = 0, \quad \frac{\delta E_L}{\delta \lambda} = 0 \]

\[ \sigma_{BC} = \sigma_{AB} + \sigma_{CA} \cos \theta \]
Shape of Droplet with Triple-Junction

\[ E_L = \pi r^2 \left[ 2\sigma_{CA} (1-z) + (\sigma_{AB} - \sigma_{BC}) (1-z^2) - r\lambda ((2+z)(1-z)^2/3-V_0) \right] \]

\[ z = \cos \theta \]

\[ \frac{\delta E_L}{\delta r} = 0: \quad \pi r \left[ 4\sigma_{CA} (1-z) + 2(\sigma_{AB} - \sigma_{BC}) (1-z^2) - r\lambda (2+z)(1-z)^2 \right] = 0 \]

\[ \frac{\delta E_L}{\delta z} = 0: \quad \pi r^2 \left[ -2\sigma_{CA} - 2(\sigma_{AB} - \sigma_{BC}) z + r\lambda (1+z)(1-z) \right] = 0 \]

\[ \sigma_{CA} z + (\sigma_{AB} - \sigma_{BC}) = 0 \]
Shape of Droplet with Triple-Junction

\[ \sigma_{BC} = \sigma_{AB} + \sigma_{CA} \cos \theta \]

\[ \sigma_{BC} > \sigma_{AB} + \sigma_{CA} \]

\( \theta \) is not defined! \( \rightarrow \) Wet
Shape of Droplet with Triple-Junction

\[ \sigma_{AB} + \sigma_{CA} > \sigma_{BC} \]

\[ \sigma_{BC} = \sigma_{AB} + \sigma_{CA}\cos\theta \]

\[ \sigma_{BC} > \sigma_{AB} \]

\[ \sigma_{BC} < \sigma_{AB} \]

\[ (\sigma_{BC} > \sigma_{AB} - \sigma_{CA}) \]
Shape of Droplet with Triple-Junction

超撥水

super-water-repellent

http://nagare-furoshiki.com/ja/repel.html
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Numer. model of Two-phase field

Problem

To represent two-phase field approximately using a function over a region
Why function?

We consider the lattice in a concerned region.
Why function?

By using functions over lattice points, cells, and edges, we represent the matters.
**Basic Approaches!**

**Level set method**

Interface = Zero crossing

**Phase field method**

\[ \xi = 0 \]
\[ \xi = 1 \]

Intermediate region
We employed the phase field method
By making the width $\varepsilon$ of the intermediate regions over several lattice units, the points whose value is 0.5 represent the interface!
The phase field method for two-phase field requires a smooth partition of unity over a continuous domain in $\mathbb{R}^3$

$$\xi(x) + \xi_c(x) = 1$$
1. We consider the phase field with intermediate region of $\varepsilon$ or a smooth partition of unity over a continuous domain in $\mathbb{R}^3$.
2. We compute the interface energy approximately.
3. By variational method, we derive an differential equation.

**Discritization**
(Restriction to a lattice points)

The difference equation of the shape, numerical model

It is not difficult to obtain a difference equation of the shape from the differential equation.
Ω: 3 dim Domain $\Omega \subset \mathbb{R}^3$

B: a connected compact set in $\Omega$

Integral Representation of Area $A$ of $\partial B$

Characteristic Function

$$\theta (x) = \begin{cases} 
1 & x \in B^0 \\
0 & x \in B^c 
\end{cases}$$

$\theta$ is differentiable as a hyperfunction.

$$A = \int_{\Omega} |\nabla \theta (x)| \, d^3x$$

$d^3x := dx^1 dx^2 dx^3$
\[
A = \int_{\Omega} |\nabla \theta(x)| \, d^3x \\
= \int_{\Omega} \left( \int_{\partial B} \partial_{q} \theta(q) \, dq \right) \, d^2s \\
= \int_{\partial B} \left( \int_{\Omega} d\theta \right) \, d^2s = \int_{\partial B} d^2s
\]
Category of the hyperfunctions

Category of the smooth functions (up to $\varepsilon$)
Integral approximation of the area of \( \partial B \) in terms of \( \xi \)-function

Filtration of closed sets (connected) \( M \subset B \subset L \) s.t.

\[ \varepsilon /2 = \text{dist}(p, B), \ p \in M, \ L^c \]

\( \xi(x) = \begin{cases} 
1 & x \in M^o \\
0 & x \in L^c 
\end{cases} \)

\( \xi: \text{smooth, monotonic increasing for normal direction} \)

\[ A_\xi = \int_\Omega |\nabla \xi(x)| \, d^3x \]

\[ |A_\xi - A| < \varepsilon \cdot K_{A_\xi} > 0 \]
\[ A_\xi = \int_{\Omega} |\nabla \xi(x)| \, d^3x \]
\[ = \int_{\Omega} \left( \int \partial_a \xi(q) \, dq \right) \, d^2s \]
\[ < \int_{\partial B} \left( 1 + \varepsilon / \min_{a=1,2} R_a(s) \right) \, d^2s \]
\[ < (1 + \varepsilon / R_{inf}) \int_{\partial B} \, d^2s \]

\( R_{inf} : \text{inf. of the principal curvature radius } R_a(s) \)
NG case:

Assume $\varepsilon \ll R_{\text{inf}} : \inf \text{ of the principal curvature radius}$
Integral approximation of the area of $\partial B$ in terms of $\xi$-function

$$\frac{\delta}{\delta \xi(x)} A_\xi = -\partial_i \frac{\partial_i \xi}{|\nabla \xi|}(x)$$

$$= \kappa(x),$$

$\kappa / 2$: approximation of mean curvature because mean curvature is given as $\text{div } \mathbf{n}/2$

for the normal unit vector field $\mathbf{n}$
We have a constant mean curvature surface approximately for constant $p_1$ and $p_2$. 

\[ \mathcal{F}_{\text{two}} := \int_{\Omega} (\sigma |\nabla \xi| - (p_1 \xi + p_2 \xi^c)) \, d^3x, \]

Approximation of Laplace equation

\[ (p_1 - p_2) - \sigma \kappa(x) = 0, \quad x \in (\text{supp}(|\nabla \xi|))^\circ \]
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Phase field model with triple-junction

To express the cone singularity in terms of phase-field method
Phase field model with triple-junction

Idea:
1. Express these regions in terms of smooth partition of unity.
2. Compute the integral approximation of the area of the interfaces!
3. We derive the Laplace equation approximately.

Problems:
- To estimate the effect from the triple-junction on the integral since it is singular.
- To show that the model is not ill-posed!
Phase field model with triple-junction

Problems:
- To estimate the effect from the triple-junction on the integral since it is singular.
- To show that the model is not ill-posed!

These problems are of the category of $C^\infty$-functions since we consider smooth partition of unity!

But it is difficult to estimate it and we use the idea of the singularity theory!
Stratification:

There is a natural stratification (滑層分割) and we consider “differentiable” submanifolds.
We express the triple-phases in terms of closed sets whose union corresponds to the domain and intersection corresponds to interface!

\[ B_a : \text{closed set} \]

\[ \Omega = B_1 \cup B_2 \cup B_3 \]

\[ B_1 \cap B_2 \subset \partial B_a, \quad a=1,2 \]

\[ B_1 \cap B_2 \cap B_3 \subset \partial (B_1 \cap B_2), \]
Stratification:

\[ B_1 \cap B_2 \cap B_3 \subset B_1 \cap B_2 \subset B_1, \]

\[ B_1 \cap B_2 \subset \partial B_a, \quad a=1,2 \]
\[ B_1 \cap B_2 \cap B_3 \subset \partial (B_1 \cap B_2), \]
Partition of Unity

\[ M_a \subset B_a \subset L_a \quad \varepsilon = \text{dist}(p, B_a) \quad p \in M_a, \overline{L_a}^c \]

\[ \xi_a (x) = \begin{cases} 
1 & x \in M_a^o \\
0 & x \in L_a^c 
\end{cases} \]

\( \xi_a \) smooth, mono.-increasing for normal dir.

\[ \xi_1(x) + \xi_2(x) + \xi_3(x) = 1 \]
\( M_a \subset B_a \subset L_a \) \( \varepsilon = \text{dist}(p, B_a) \) \( p \in M_a, \overline{L_a}^c \)

\[ \xi_a(x) = \begin{cases} 
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\[
\xi_a(x) = \begin{cases} 
1 & x \in M_a^o \\
0 & x \in L_a^c
\end{cases}
\]

\( \xi_a \) smooth, mono.-increasing for normal dir.

\[ \xi_1(x) + \xi_2(x) + \xi_3(x) = 1 \]
Integral Approximation of the area in terms of the Partition of Unity

\[ A_{ab}^\xi = \int_{\Omega} (|\nabla \xi_a| |\nabla \xi_b|)^{1/2}(\xi_a + \xi_b) \, d^3x \]

Surface Free Energy

\[ F(\xi) = \int_{\Omega} (\sum_{a,b} \sigma_{ab} (|\nabla \xi_a| |\nabla \xi_b|)^{1/2}(\xi_a + \xi_b) - p(x)) \, d^3x \]

\[ p(x) = \sum p_a \xi_a(x) \]
We derive the equation using the stratified structure of this system!
We estimate the accuracy of this model each part:

1. \( x \in \Omega, \text{ s.t. } \xi_a(x) = 1, \)

2. \( x \in \text{supp}(\xi_a) \cap \text{supp}(\xi_b) = L_a \cap L_b \)
   \( \text{ s.t. } \xi_a(x) + \xi_b(x) = 1, \)

3. \( \text{supp}(\xi_1) \cap \text{supp}(\xi_2) \cap \text{supp}(\xi_3) = L_0 \cap L_1 \cap L_2 \)
Results of variation method

$$\frac{\delta}{\delta \xi} F[\xi] = 0$$

for each region:

1. $x \in \Omega$, $\xi_a(x) = 1$

$0 = 0$
We show that this equation does not have bad effects on the model!

2. \( x \in \text{supp}(\xi_a) \cap \text{supp}(\xi_b) \) s.t. \( \xi_a(x) + \xi_b(x) = 1 \)

\[
(p_a - p_b) - \sigma_{ab} \kappa_a(x) = 0. \quad \kappa_a := -\partial_i \frac{\partial_i \xi_a}{|\nabla \xi_a|}.
\]

\( \rightarrow \) reproduces the interface of two-phase field

3. \( x \in \text{supp}(\xi_1) \cap \text{supp}(\xi_2) \cap \text{supp}(\xi_3) \)

\[
(p_a - p_b - p_c) - \tilde{\kappa}_{abc}(x) = 0,
\]

\[
\tilde{\kappa}_{abc} := \sigma_{bc} \sqrt{|\nabla \xi_b(x)||\nabla \xi_c(x)|} - \sigma_{ab} \sqrt{|\nabla \xi_a(x)||\nabla \xi_b(x)|} - \sigma_{ac} \sqrt{|\nabla \xi_a(x)||\nabla \xi_c(x)|}
\]

\[
+ \partial_i \left[ \frac{\partial_i \xi_a}{|\nabla \xi_a|^3} \cdot \left( \sigma_{ab} \sqrt{|\nabla \xi_b|}(\xi_a + \xi_b) + \sigma_{ac} \sqrt{|\nabla \xi_c|}(\xi_a + \xi_c) \right) \right].
\]
We evaluate the energy and consider it weakly up to $\epsilon$!

1. $x \in \Omega$, $\xi_a(x) = 1$,

2. $x \in \text{supp}(\xi_a) \cap \text{supp}(\xi_b) = L_a \cap L_b$
   s.t. $\xi_a(x) + \xi_b(x) = 1$,

3. $\text{supp}(\xi_1) \cap \text{supp}(\xi_2) \cap \text{supp}(\xi_3) = L_0 \cap L_1 \cap L_2$
Evaluate the surface energy!

\[ E(\Box) = \int \left( \sum_{a,b} \sigma_{ab} \left( |\nabla \xi_a| |\nabla \xi_b| \right)^{1/2} (\xi_a + \xi_b) \right) d^3x \]
Evaluate the surface energy!

\[ E(\Box) = \int (\sum_{a,b} \sigma_{ab} (|\nabla \xi_a| |\nabla \xi_b|)^{1/2} (\xi_a + \xi_b)) d^3x \]

Then we have the following estimations:

1. \( x \in \Omega, \xi_a(x) = 1, \)
   \[ E(\{x \in \Omega, \xi_a(x) = 1\}) = 0 \]
Then we have the following estimations:

2. \( x \in \text{supp}(\xi_a) \cap \text{supp}(\xi_b) = L_a \cap L_b \)
   
   \[ \text{s.t. } \xi_a(x) + \xi_b(x) = 1, \]

\[ |E(\text{supp}(\xi_1) \cap \text{supp}(\xi_2)) - \sigma_{12} A_{12}| < \varepsilon K \]

For \( \varepsilon \) to 0, the model gives the energy well.
Evaluate the surface energy!

\[
E(\square) = \int_{\square} \left( \sum_{a,b} \sigma_{ab} |\nabla \xi_a| |\nabla \xi_b| \right)^{1/2} (\xi_a + \xi_b) d^3x
\]

Then we have the following estimations:

3. \(\text{supp}(\xi_1) \cap \text{supp}(\xi_2) \cap \text{supp}(\xi_3) = L_0 \cap L_1 \cap L_2\)

\[
E(\text{supp}(\xi_1) \cap \text{supp}(\xi_2) \cap \text{supp}(\xi_3)) < \varepsilon K'
\]

For \(\varepsilon\) to 0, the model gives the energy well.
By applying the idea of the stratification to this model, we can estimate the accuracy of this model.

We conclude that this model approximates the triple-junction well up to $\varepsilon$!
Using the model (via the difference Euler equation with viscous force term), the surface tension values reproduce the expected contact angle!
Reduced Mathematical Problem

To construct a mathematical model in order to evaluate the triple junctions numerically!

Mathematical Problem

To construct a mathematical model in order to evaluate fluid dynamics with triple-phase and triple junctions numerically!
By revising the mathematical model of two-phase fluid dynamics to obtain that of three-phase field!

Discretization

2-phase Euler Equation → 2-phase Difference Euler Equation

Discretization

3-phase Euler Equation → 3-phase Difference Euler Equation

Applied Math (VOF-method)
Arnold, Ebin-Marsden 1970

Diffeomorphism Group +Momentum Map

Variation

Discretization

1-phase Action Functional
1-phase Euler Equation
1-phase Difference Euler Equation

2-phase Action Functional
2-phase Euler Equation
2-phase Difference Euler Equation

3-phase Action Functional
3-phase Euler Equation
3-phase Difference Euler Equation

Analytic Geometry

Singularity Theory

Applied Math (phase-field theory)

Applied Math (VOF-method)
We can compute the three-phase fluid dynamics with triple-junction numerically!

Meniscus Motion

We can compute the three-phase fluid dynamics with triple-junction numerically!

Numerical computation: emission of ink droplets

Emission device of ink droplets

Several x 10 um
Thank you for your attention!
Modified Laplace equation:

\[ \sigma \left( \sum_j \partial_i \frac{\partial_j \xi \partial_j \xi}{|\nabla \xi|} - \sum_j \partial_j \frac{\partial_j \xi \partial_i \xi}{|\nabla \xi|} \right) - (p_1 - p_2) \partial_i \xi = 0, \]

or

\[ \partial_j \tau_{ij}(x) - (p_1 - p_2) \partial_i \xi(x) = 0, \]

where

\[ \tau(x) := \sigma \left( I - \frac{\nabla \xi}{|\nabla \xi|} \otimes \frac{\nabla \xi}{|\nabla \xi|} \right) |\nabla \xi|(x). \]
Modified Laplace equation:

\[
\partial_i p_P - \sum_{a > b} \sigma_{a,b} \left[ \partial_i \left( \frac{\sqrt{|\nabla \xi_a| |\nabla \xi_b|} (\xi_a + \xi_b)}{\sqrt{|\nabla \xi_a|^3}} \right) \right] = 0,
\]

\[p_P(x) := \sum p_a \xi_a(x), \quad x \in \Omega.\]
$\varepsilon$-tubular neighbourhood

$T_{U, \varepsilon}$ of $U$
phase field法による2相の流体方程式（オイラー方程式）:

\[ S_{\text{two}}[\gamma] = \int_T dt \int_\Omega \left( \frac{1}{2} \rho |u|^2 - \sigma |\nabla \xi| + (p_1 \xi + p_2 \xi^c) \right) d^3x. \]

変分原理

\[ \frac{D\rho u^i}{Dt} + \sigma \left( \sum_j \partial_i \frac{\partial_j \xi \partial_j \xi}{|\nabla \xi|} - \sum_j \partial_j \frac{\partial_j \xi \partial_i \xi}{|\nabla \xi|} \right) + \partial_i p = 0. \]
**Basic Approaches!**

**Level set method**

Interface = Zero crossing

**Phase field method**

Intermediate region
Level Set method

Zero points of signed distant function express the interface!
⇔ The ridge appears inside!
Its singularity causes difficulty.

Interface = Zero crossing
By making the width $\varepsilon$ of the intermediate regions over several lattice units, the points whose value is 0.5 represent the interface!
We employed the phase field method for two-phase problems. The phase field method requires a smooth partition of unity over a continuous domain in $\mathbb{R}^3$.

$$\xi(x) + \xi^c(x) = 1$$
Deviation from today’s topic

Singularity of light ray = Caustics

Michel Berry

Catastrophe theory and Caustics 1970’s.

Orbits from an electron emission device for an display cause caustics. (1998)
Though it has a bad effect on phosphor (蛍光体), it cannot be avoided due to the stability. We could concentrate ourselves on another countermeasure.

Singularity theory is of use for many situations in industrial studies besides inkjet printer!
\[ A_\xi = \int |\nabla \xi(x)| \, d^3x \]
\[ = \int \left( \int_{\partial B} \partial_a \xi(q) \, dq \right) \, d^2s \]
\[ < \int_{\partial B} \left| 1 + \frac{\varepsilon}{\min R_a(s)} \right| \, d^2s \]
\[ < (1 + \frac{\varepsilon}{R_{\text{inf}}}) \int_{\partial B} \frac{d^2s}{d} \]

\[ q(x) = 0 \]

\[ q \]

\[ S_2 \]

\[ S_1 \]
Example: ball with $R$ (but not smooth $\xi$)

$\xi(r) = \begin{cases} 
1, & r < R - \varepsilon/2, \\
0, & r > R + \varepsilon/2, \\
\frac{(r-R)}{\varepsilon}, & \text{otherwise}
\end{cases}$

$A_\xi = \int |\nabla \xi(x)| \, d^3x$

$= \int_{-\varepsilon/2}^{\varepsilon/2} 4\pi (R+q)^2/\varepsilon \, dq$

$= 4\pi/\varepsilon \left[ R^2q + Rq^2 + q^3/3 \right]_{-\varepsilon/2}^{\varepsilon/2}$

$= 4\pi R^2 + \pi \varepsilon^2/3$
NG case:

\[ \epsilon \ll R_{\text{inf}} : \text{inf of the principal curvature radius} \]